Solow model

Assumptions of Solow model

- Output is produced by neoclassical production function $Y = F(K, L)$ that has usual properties (diminishing marginal product, constant returns to scale)
- The economy is closed, savings are equal to investments, $S \equiv I$,
- Resource constraint is then given by $Y = C + I = C + S$
- Savings are constant share of income, $S = sY$, consumption is then $C = (1-s)Y$
- Population growth is given exogenously (outside of the model); it growths at rate $\frac{\Delta L}{L} = n$
- Law of motion for capital is given by equation: $K_{t+1} = (1-\delta)K_t + I_t$ or equivalently $\Delta K = I_t - \delta K_t$

Intensive form of production function

- The production function is homogenous of degree one (which is equvalent to CRS). It holds:

$$F(cK, cL) = cF(K, L)$$

- We can premultiply the production function by $\frac{1}{L}$ to get

$$F\left(\frac{K}{L}, \frac{L}{L}\right) = F\left(\frac{K}{L}, 1\right) = \frac{1}{L}F\left(K, L\right) = \frac{Y}{L}$$

  and using definition $k = \frac{K}{L}$ (capital per capita, or capital intensity of labor) and $y = \frac{Y}{L}$ (output per capita) we get

$$F\left(\frac{K}{L}, 1\right) = \frac{Y}{L} \Rightarrow f(k) = y$$

  which is so called intensive form of production function, which maintain usual properties of the original function.

Solution of the model

- We are interested in behaviour of capital in time Δk. The definition of capital intensity $k = \frac{K}{L}$ is compound function (of K and L), we can calculate its total differential:

$$\Delta k = \frac{1}{L}\Delta K - \frac{K}{L^2}\Delta L$$

  and use some previous definitons

$$\Delta k = \frac{I - \delta K}{L} - \frac{K}{L}\frac{\Delta L}{L}$$

$$\Delta k = \frac{sY}{L} - \delta\frac{K}{L} - n\frac{K}{L}$$

$$\Delta k = sy - \delta k - nk$$

- We have got fundamental equation for the model that governs its dynamic:

$$\Delta k = sf(k) - (\delta + n)k \quad (1)$$

- The term $sf(k)$ denotes gross investment, the term $(\delta + n)k$ denotes investment that goes to replacement of depreciated capital δk and to eqiupment of new workers with capital nk.

- This behaviour can be desribed by phase diagram. The system converges towards the state where $\Delta k = 0$. We call this *steady-state*.

- The steady state level of capital per worker k^* is determined by:

$$sf(k^*) = (n + \delta)k^*$$

  which gives constant level of production per capita:

$$y^* = f(k^*)$$

- Thus in the long run, when the economy has converged to the steady state, there is *no growth in production per capita*. Thus the model *can not explain perpetual growth in production per capita.*

- In the steady state Y, K and L all grow at the same rate n. Hence we are on a balanced growth path.
- Increase in the population growth will decrease *level* of output per capita but leave its growth rate in the long run unchanged.
- The role of the savings rate: An increased savings rate increases the *level* of output per capita in the long run but not the growth in the long run.
- Since the savings rate is bounded above by 1, continually increasing the saving rate can not give perpetual growth.
- Policy can not affect growth in long run.

The transitional dynamics

- It is illustrative to consider the dynamics in a (k, γ_k) diagram, that is, where we can read off the growth rate $\gamma_k = \Delta k/k$ vertically.
- We plot the transformed version of the equation (1).

$$\frac{\Delta k}{k} = s\frac{f(k)}{k} - (n + \delta)$$

  where $f(k)/k$ is the average productivity of capital, which is declining in k.
- This illustrates a very important implication of the model: If the two countries have the same steady states, the poorer country will grow faster.

Technological progress

- We can include exogenous technological progress into the model. Since this progress is unexplained, we do not learn much new about the sources of growth. But it is a useful exercise because we 1) can use the model to do growth accounting, 2) see how technological progress affects the dynamics.
- We rewrite the production function

$$Y(t) = F(K(t), L(t), A(t))$$

  where $A(t)$ is a shift parameter that captures technological progress. $A(t)$ grows at a constant rate $\gamma_A = \Delta A/A = x$.

- It can be shown that technological progress must be modeled as labor augmenting, i.e.

$$Y(t) = F(K(t), A(t)L(t)) \tag{2}$$

- We now instead of dividing all quantities by the stock of labor L, divide by the stock of labor measured in efficiency units, AL. (Define $\hat{k} \equiv K/AL$, $\hat{y} \equiv Y/AL$.)

  The model is structurally the same as before, the only change is that the term n is now replaced by $n + x$.

- Thus we reach a steady state $\hat{k}^*$ as before and hence also a steady state level of production per effective worker, $\hat{y}^*$. On the balanced growth path we thus have

$$\gamma_{\hat{y}} = \gamma_{Y/TL} = 0 \Rightarrow \gamma_{Y/L} = \gamma_A = x$$

  i.e. GDP per capita grows at the rate of exogenous technological change (x).

This implies:

- Growth in GDP per capita in the long run is due to unexplained technological progress

Further readings

Burda, M., Wyplozs, C. *Macroeconomics: A European Text,* 2001, Chapter 3.